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Quantizing the de Sitter space–times

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Abstract

In this Letter, the cosmological horizons are quantized by the reduced phase space approach [Phys. Lett. B 517 (2001) 415], in the cases of Schwarzschild–de Sitter and pure de Sitter space–times. The same discrete spectrum is obtained, either the mass or the cosmological constant plays the role of dynamical variable. We also briefly discuss the possible relation between the discrete spectrum and the *N bound* of holographic principle.

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1. Introduction

Although a perfect theory of quantum gravity is developing, some well-established predictions, such as black hole thermodynamics [2], have been obtained from the quantum field theory in curved space–time. On the other hand, Bekenstein has also proposed that the area spectrum of a black hole is discrete and uniformly spaced [3]. Bekenstein's proposal is based on the observation that the horizon area of black hole is an adiabatic invariant [4]. According to Erenfest principle [5] of the old quantum theory, an adiabatic invariant corresponds to a quantum number. For example, the Jacobi integral satisfies the Sommerfeld's quantization condition $I = \oint p\,dq = 2\pi n\hbar$. Many efforts [6–17] have been devoted to the quantization of black holes, and Bekenstein's spectrum has been rediscovered. Especially, a method called "reduced phase space quantization" [1,17] looks simple and elegant. Furthermore, it is compatible with the algebraic approach [16,17]. In this model, some coordinate invariants, such as black hole's mass M and charge Q, are treated as the dynamical variables of the system [6]. The phase space is constructed by these variables and their conjugate momenta. Due to the coordinate invari-

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ance of these variables $\{Z_i\}$, the reduced action must be of the form [1,6]

$$I^{\text{red}} = \int dt \left[\sum_i P_{Z_i} \dot{Z}_i - H(Z_i) \right], \tag{1}$$

which describes the dynamics of the static and spherically symmetric configurations in any classical theory of gravity. The specific expression of the Hamiltonian is irrelevant except to be independent of P_{Z_i}. In the example of a Schwarzschild black hole, M is viewed as the dynamical variable and P_M is its conjugate momentum. The latter can be understood as the difference between the time at either end of the space-like slice. We can find a transformation $(M, P_M) \to (X, \Pi_X)$, and the area of black hole is written as $A \sim \frac{1}{2}(X^2 + \Pi_X^2)$. Performing the standard quantization, we obtain a discrete area spectrum, $A = \alpha(n + \frac{1}{2})$, α is of order of Planck area l_{p}^2. This method will be utilized to quantize the de Sitter space–time in this Letter.

The aim of this Letter is partly motivated by the similarities between black holes and de Sitter space–time. Gibbons and Hawking have also found that there is thermal radiation from the cosmological horizon [18]. The temperature and entropy are proportional to the surface gravity κ and the area A of cosmological horizon, respectively. Can we quantize the de Sitter space–time in a similar manner?

The aim of this Letter is mainly motivated by the evidences that the present universe may be driven by a positive cosmological constant,[1] $\Lambda \sim H_0^2$, H_0 is the present Hubble parameter. The cosmological constant is closely related to the density of vacuum energy. However, the observed cosmological constant is some 120 orders less than its natural value, $\Lambda/m_{\mathrm{p}}^2 \sim 10^{-120}$. The discrepancy is a serious challenge to theoretical physics. Although it is far from settling this problem, many efforts, such as Euclidean quantum gravity [20] and holographic principle [21,22], have been devoted to it. It is generally believed that the solution to the cosmological constant problem depends on a full quantum theory of gravity. The efforts along the roads to quantum gravity are helpful. We hope to get an insight into the possible relation between holographic principle and the quantization of de Sitter space–time.

[1] There are some reasons to consider dynamical dark energy as an alternative to Λ, such as quintessence and phantom, etc. However, the constraints on the dark energy equation of state parameter do not exclude the cosmological constant. For details, see the review [19] and references therein.

2. Quantization

2.1. *M as a dynamical variable*

According to the reduced phase space method, we need to find the dynamical variables appropriate for the quantization of de Sitter space–time. In the cases of black holes, one of the dynamical variables is the mass M. We first consider an asymptotic de Sitter space–time as follows

$$\begin{aligned} ds^2 = {} & -\left(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\right) dt^2 \\ & + \left(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\right)^{-1} dr^2 + r^2\, d\Omega, \end{aligned} \tag{2}$$

where Λ is the cosmological constant. M is regarded as the only dynamical variable. Here we assume that there is not a black hole but an usual star in the Schwarzschild–de Sitter space–time, all the discussions are performed for the cosmological horizon.

The mass variable can be written as

$$M = \frac{1}{2}\left(r_c - \frac{\Lambda}{3} r_c^3\right), \tag{3}$$

where the cosmological horizon r_c is located by the following equation

$$1 - \frac{2M}{r_c} - \frac{1}{3}\Lambda r_c^2 = 0. \tag{4}$$

The surface gravity at the horizon is given by

$$\begin{aligned} \kappa_c &= -\frac{1}{2}\left(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\right)'_{r=r_c} \\ &= \frac{1}{2}\Lambda r_c - \frac{1}{2r_c}. \end{aligned} \tag{5}$$

So we obtain

$$dM = -\frac{\kappa_c}{8\pi} dA_c, \tag{6}$$

where $A_c = 4\pi r_c^2$ is the area of cosmological horizon. Note that there is an additional minus sign in Eq. (6), compared to the black hole thermodynamics. Since the conjugate momentum P_M plays the role of "time", we consider the following transformation

$$X = \sqrt{B(M)}\cos(k_c P_M),$$
$$\Pi_X = -\sqrt{B(M)}\sin(k_c P_M), \tag{7}$$

which naturally incorporates the periodicity of P_M, motivated by the Euclidean quantum gravity. The period is the inverse Hawking temperature, $2\pi/\kappa_c$, κ_c is a function of M. Eq. (7) is similar to the case of black hole [1]. However, there is also a minus sign in the second equation in (7). This corresponds to the minus sign in (6) and ensures that the transformation is canonical. From (7), direct calculation reveals that

$$\delta X = \frac{1}{2\sqrt{B}}\cos(\kappa_c P_M)B'\delta M - \sqrt{B}\sin(\kappa_c P_M)(\kappa_c \delta P_M + P_M \kappa_c' \delta M), \tag{8}$$

where $B' = dB/dM$, $\kappa_c' = d\kappa_c/dM$. We obtain

$$P_M \delta M - \Pi_X \delta X = \eta_1 \delta P_M + \eta_2 \delta M, \tag{9}$$

where

$$\eta_1 = -B\sin^2(\kappa_c P_M)\kappa_c,$$
$$\eta_2 = P_M + \frac{1}{2}B'\sin(\kappa_c P_M)\cos(\kappa_c P_M) - B\sin^2(\kappa_c P_M)P_M \kappa_c'. \tag{10}$$

The transformation $(M, P_M) \to (X, \Pi_X)$ is canonical if and only if $\partial\eta_1/\partial M = \partial\eta_2/\partial P_M$, i.e.,

$$1 + \frac{1}{2}B'\kappa_c = 0, \tag{11}$$

where two terms consisting of κ_c' cancel each other out. Comparing (11) with (6), we have

$$B = \frac{A_c}{4\pi} = X^2 + \Pi_X^2, \tag{12}$$

which is similar to the Hamiltonian of a harmonic oscillator. Following the standard procedures of quantization, we substitute the operators for X and Π_X, and then obtain the discrete spectrum of the cosmological horizon

$$A_c = 8\pi\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots, \tag{13}$$

where the quantization condition $[\hat{X}, \hat{\Pi}_X] = i$ has been imposed on (12). We have also defined the quantum number operator as $\hat{n} = \hat{a}^\dagger\hat{a}$, $[\hat{a}, \hat{a}^\dagger] = 1$, where

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{\Pi}_X),$$
$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{X} + i\hat{\Pi}_X). \tag{14}$$

The area of cosmological horizon can be expressed by the Jacobi integral as follows

$$A_c = -4\oint \Pi_X\, dX = \kappa_c A_c \int_0^{2\pi/\kappa_c} \sin^2(\kappa_c P_M)\, dP_M. \tag{15}$$

The discrete spectrum is also obtained by imposing the Sommerfeld's quantization condition on the Jacobi integral.

2.2. *Λ as a dynamical variable*

The discrete area spectrum of a pure de Sitter space–time can be obtained by setting $M = 0$, after quantizing the Schwarzschild–de Sitter space–time (2). Can we quantize the pure de Sitter space–time in a direct manner? The mass variable does not appear in the pure de Sitter space–time naturally, so we need to find a new dynamical variable. Since the cosmological constant is the only parameter in the pure de Sitter space–time, we naturally treat it as the dynamical variable.[2] The dynamics of the pure de Sitter space–time can be described by the reduced action

$$I^{\text{red}} = \int dt\,[P_\Lambda \dot{\Lambda} - H(\Lambda)], \tag{16}$$

which follows from the argument of Ref. [1] that the coordinate invariants can be viewed as the dynamical variables, and ensure that the Hamiltonian is independent of the conjugate momenta. On the other hand, we

[2] We can define a phenomenological energy as $M_{\text{vac}} = r_c/2$, according to the thermodynamics of de Sitter space–time. Following the procedures in Section 2, we obtain the same spectrum again. However, it is more natural that Λ is viewed as the dynamical variable.

have the following relation

$$-V\,d\Lambda = \kappa\,dA_c, \tag{17}$$

where κ is the surface gravity at the cosmological horizon, V is the spatial volume within the cosmological horizon, $V = 4\pi r_c^3/3$. We consider the following transformation

$$X = \sqrt{\frac{A_c(\Lambda)}{4\pi}}\cos\big[Q(\Lambda)P_\Lambda\big],$$
$$\Pi_X = -\sqrt{\frac{A_c(\Lambda)}{4\pi}}\sin\big[Q(\Lambda)P_\Lambda\big], \tag{18}$$

where $Q(\Lambda)$ is to be determined, and it imposes a periodic boundary condition on the phase space constructed by $\{\Lambda, P_\Lambda\}$. We obtain

$$\begin{aligned} &P_\Lambda\delta\Lambda - \Pi_X\delta X \\ &\quad= \bigg[P_\Lambda + \frac{1}{8\pi}A_c'\sin(QP_\Lambda)\cos(QP_\Lambda) \\ &\qquad - \frac{1}{4\pi}P_\Lambda Q' A_c\sin^2(QP_\Lambda)\bigg]\delta\Lambda \\ &\qquad - \frac{A_c}{4\pi}Q\sin^2(QP_\Lambda)\delta P_\Lambda, \end{aligned} \tag{19}$$

where $Q' = dQ/d\Lambda$, $A_c' = dA_c/d\Lambda$. When the following equation

$$1 + \frac{Q}{8\pi}A_c' = 0 \tag{20}$$

is satisfied, the transformation (18) is canonical. Comparing (20) with (17), we have $Q = 8\pi\kappa/V$, which is in inverse proportion to the 4-volume of Euclidean–de Sitter space. The cosmological horizon area is rewritten as

$$A_c = 4\pi\big(X^2 + \Pi_X^2\big), \tag{21}$$

which is the same as (12). Imposing the quantization condition $[\hat{X}, \hat{\Pi}_X] = i$ on it, we obtain the area spectrum (13) again.

3. Discussions

We make some remarks about the discrete spectrum of de Sitter space–time.

(a) Banks [24] has proposed that the universe with $\Lambda > 0$ is described by a quantum theory of gravity with the finite number of degrees of freedom (d.o.f.)

$$N = \frac{3\pi}{\Lambda}, \tag{22}$$

which is just the entropy of the pure de Sitter space–time. Bousso [25] therefore argues that a universe with a positive cosmological constant cannot have entropy greater than $N = 3\pi/\Lambda$, named *N bound*. N bound should be attributed to the d.o.f. of the quantum gravity. The discrete spectrum (13) gives a relation between the number of d.o.f. and the quantum number n. From (13), we have

$$2n + 1 = \frac{3}{\Lambda}, \quad n = 0, 1, 2, \ldots. \tag{23}$$

Comparing it with the N bound, we have $N = 2n + 1$. Since the de Sitter space–time has a finite horizon, the corresponding quantum number is also finite. The *finite number* of d.o.f. can be attributed to the discrete spectrum of de Sitter space–time.

(b) If n is the only quantum number, we can construct a n-dimensional Hilbert space spanned by the area eigenstates. A entropy bound is naturally given by [26]

$$S(n) = \ln n \sim \ln A_c, \tag{24}$$

which is far from matching the Bekenstein–Hawking entropy. The quantum number n is insufficient to construct a complete set to describe the full quantum states of de Sitter space–time. This implies that there exist other unknown quantum numbers (or new degrees of freedom) that make the area eigenvalue exponentially degenerate. The degrees of degeneracy must be $g(n) = \exp(2n+1)$, in order to obtain the Bekenstein–Hawking entropy. The similar degeneracy has been proposed by Mukharnov and Bekenstein [23], in the case of discrete black hole area.

(c) We have been taught that a natural cosmological constant (or vacuum energy density) is given by summing over all modes with zero point energy[3]

$$\rho_{\text{vac}} \sim \int^{m_{\text{p}}} d^3k\,\sqrt{k^2 + m^2} \sim m_{\text{p}}^4,$$
$$\Lambda = 8\pi G\rho_{\text{vac}} \sim m_{\text{p}}^2, \tag{25}$$

[3] Let us temporarily forget Banks' philosophy that Λ is viewed as an input parameter [24].

where m_{p} is the Planck mass and plays the role of UV cutoff. Such a result can also be obtained from the discrete spectrum (13), if $n \sim 0$. However, we have no reason to prefer such a small number. On the contrary, our universe has a large number of d.o.f., which implies a small cosmological constant.

It is somewhat surprise that the cosmological constant is in inverse proportion to the number of d.o.f. Padmanabhan's proposal [27] may be helpful to understanding this extraordinary feature, at a qualitative level. He argues that the observed cosmological constant is the residual quantum fluctuations as the tiny part of the vacuum energy. Thus we can intuitively understand why Λ decreases with the increasing number of degrees of freedom.[4] However, we are not clear how the quantum structure of space–time absorbs the huge vacuum energy. The fine tuning problem remains to be solved.

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[4] This is an analogy with the canonical ensemble: the thermal fluctuations in energy also decrease with the increasing number of particles.

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